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On the quiver of tilting modules [☆]

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Let Λ be an Artin algebra over a commutative Artin ring R and let $\operatorname{mod}\Lambda$ be the category of finitely generated left Λ-modules. For a module $M \in \operatorname{mod}\Lambda$, we denote by $\operatorname{pd}_\Lambda M$ the projective dimension of M and by $\operatorname{gl.dim}\Lambda$ the global dimension of Λ. A module $T \in \operatorname{mod}\Lambda$ is called a tilting module provided the following three conditions are satisfied:

(i) $\operatorname{pd}_\Lambda T < \infty$,
(ii) $\operatorname{Ext}^i_\Lambda(T, T) = 0$ for all $i > 0$, and
(iii) there exists an exact sequence $0 \to {}_\Lambda\Lambda \to T^0 \to \cdots \to T^r \to 0$ with $T^i \in \operatorname{add} T$ for all $0 \leqslant i \leqslant r$, where $\operatorname{add} T$ is the full subcategory of $\operatorname{mod}\Lambda$ whose objects are direct sums of direct summands of T.

We will say that a tilting module is basic if in a direct sum decomposition of T the indecomposable summands of T occur with multiplicity one. Unless stated otherwise all tilting modules considered here will be assumed to be basic.

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Following [AR] we consider for a tilting module the right perpendicular category

$$T^{\perp} = \left\{ X \in \operatorname{mod} \Lambda \;\middle|\; \operatorname{Ext}^i_{\Lambda}(T, X) = 0 \text{ for all } i > 0 \right\}.$$

Important properties of these categories will be recalled in Section 1. We consider the set $\mathcal{T}_\Lambda$ of all tilting modules over Λ up to isomorphism. In [HU4] the following partial order $\leqslant$ on $\mathcal{T}_\Lambda$ was investigated. For $T, T' \in \mathcal{T}_\Lambda$, we set $T \leqslant T'$ provided $T^{\perp} \subset T'^{\perp}$. One of the main results in [HU4] was that the Hasse quiver $\vec{\mathcal{K}}_\Lambda$ of this poset allows the following description. The vertices of $\vec{\mathcal{K}}_\Lambda$ are the elements of $\mathcal{T}_\Lambda$. There is an arrow $T' \to T$ if $T' = M \oplus X$, $T = M \oplus Y$ with X, Y indecomposable, and there exists a short exact sequence $0 \to X \to \widetilde{M} \to Y \to 0$ with $\widetilde{M} \in \operatorname{add} M$. In Section 2 we will recall the reason behind this. Moreover, we will show the following for a representation-finite algebra Λ of finite global dimension d: *For each* $0 \leqslant i \leqslant d$ *there exists a tilting module* $T \in \vec{\mathcal{K}}_\Lambda$ *with* $\operatorname{pd}_\Lambda T = i$.

The main aim of this article is to investigate the local structure of $\vec{\mathcal{K}}_\Lambda$ in more detail. In particular, we will establish a precise relationship between the number of neighbours of a vertex $T \in \vec{\mathcal{K}}_\Lambda$ and the $\operatorname{add} T$-coresolution of ${}_\Lambda\Lambda$ in the definition of a tilting module. We will also discuss the number and length of paths leading or starting in a given vertex. It is interesting to point out that the finiteness of certain paths in $\vec{\mathcal{K}}_\Lambda$ starting in a vertex T is related to the generalised Nakayama conjecture. For details we refer to Section 4. In the special case of hereditary algebras we can say more. In this case we say that a vertex $T \in \vec{\mathcal{K}}_\Lambda$ is saturated if the number of neighbours coincides with the number n of distinct simple modules. It is easy to see that the number of neighbours is always bounded by n. We will show that a vertex T is saturated if and only if each coordinate of the dimension vector of T is at least two. The results for hereditary algebras are contained in Section 3.

We denote the composition of morphisms $f : X \to Y$ and $g : Y \to Z$ in a given category $\mathcal{K}$ by fg.

The notation and terminology introduced here will be fixed throughout this article. For unexplained representation-theoretic terminology, we refer to [R] or [ARS].

1. Preliminaries

In this section we will briefly recall the basic definitions and results from tilting theory we will use in the main part of this article.

Given any Λ-module Z, we may decompose it: $Z \simeq \bigoplus_{i=1}^m Z_i^{d_i}$, where Z_i is indecomposable and $d_i > 0$ for all i and moreover $Z_i \not\simeq Z_j$ for $i \neq j$. We call Z basic in case $d_i = 1$ for all i. The uniquely determined number m occurring in the direct sum decomposition above will be denoted by $\delta(Z)$. It is not hard to see that a tilting module T satisfies $\delta(T) = n = \operatorname{rk} K_0(\Lambda)$, where $K_0(\Lambda)$ denotes the Grothendieck group of Λ. We say that a module M is a partial tilting module if M is a direct summand of a tilting module. A partial tilting module M with $\delta(M) = n - 1$ is called an almost complete partial tilting module. Given a partial tilting module M, we say that C is a complement to M if $M \oplus C$ is a tilting module and $\operatorname{add} M \cap \operatorname{add} C = 0$. Note that a complement to an almost complete partial tilting module is indecomposable. We will see later that special complements are important.

Let M be a partial tilting module and C be a complement to M, then C is called a source complement if

$$(M \oplus C)^{\perp} = M^{\perp}.$$

We refer to [HU1,HU3] for a characterisation and the concept of sink complements.

More generally, we call $X \in \operatorname{mod} \Lambda$ self-orthogonal if $\operatorname{Ext}^i_\Lambda(X, X) = 0$ for all $i > 0$, and we call X exceptional if X is self-orthogonal and satisfies $\operatorname{pd}_\Lambda X < \infty$. For more details, we refer to [H]. So partial tilting modules are typical examples of exceptional modules.

Let $T = \bigoplus_{j=1}^n T_j$ be a tilting module over Λ. For $1 \leqslant i \leqslant n$, we define $T[i] = \bigoplus_{j \neq i} T_j$. Note that $T[i]$ is an almost complete partial tilting module.

For $X \in \operatorname{mod} \Lambda$, we denote by $\operatorname{fac} X$ the full subcategory of $\operatorname{mod} \Lambda$ consisting of those modules Y which are epimorphic images of modules in $\operatorname{add} X$. Dually, we denote by $\operatorname{sub} X$ the full subcategory of $\operatorname{mod} \Lambda$ consisting of submodules of modules in $\operatorname{add} X$.

Let ${}_\Lambda T$ be a tilting module and let $X \in T^{\perp}$. As shown, for example, in [CHU], we have that $X \in \operatorname{fac} T$. So there is a short exact sequence $0 \to K_0 \to T_0 \to X \to 0$ with $T_0 \in \operatorname{add} T$ and $K_0 \in T^{\perp}$. Inductively, we obtain a long exact sequence

$$\cdots \to T_r \xrightarrow{f_r} T_{r-1} \to \cdots \to T_0 \xrightarrow{f_0} X \to 0 \qquad (*)$$

with $T_i \in \operatorname{add} T$ and $K_i = \ker f_i \in T^{\perp}$ for all $i \geqslant 0$. We call $(*)$ an $\operatorname{add} T$-resolution of X, and if $(*)$ is minimal, a minimal $\operatorname{add} T$-resolution. If $\operatorname{pd}_\Lambda X < \infty$, then X admits a finite $\operatorname{add} T$-resolution $0 \to T_r \to T_{r-1} \to \cdots \to T_0 \to X \to 0$. The following lemma is shown in [HU4] and will be needed in Section 3.

Lemma 1.1. *Let ${}_\Lambda T$ be a tilting module. Let $X \in T^{\perp}$ be an exceptional module with $X \notin \operatorname{add} T$. If $0 \to T_r \to T_{r-1} \to \cdots \to T_0 \to X \to 0$ is a minimal $\operatorname{add} T$-resolution of X, then* $\operatorname{add} T_r \cap \operatorname{add} T_0 = 0$.

We recall from [HU3] the structure of complements to almost complete partial tilting modules. This will be needed in section three. We first recall the following proposition.

Proposition 1.2. *Let M be an almost complete tilting module. Let Y be an indecomposable complement generated by M. Then*

(1) *M is faithful*;
(2) *there exists an indecomposable complement X not isomorphic to Y*;
(3) *there exists an exact sequence $0 \to X \to E \to Y \to 0$ with $E \in \operatorname{add} M$*;
(4) $\operatorname{Ext}^i_\Lambda(X, Y) = 0$ *for $i > 0$ and* $\operatorname{Ext}^i_\Lambda(Y, X) = 0$ *for $i > 1$; and*
(5) *X is uniquely determined by the property* (3).

We also point out that Y is uniquely determined by the property (3). Note that in (3) E is the minimal right $\operatorname{add} M$-approximation of Y, as well as the minimal left $\operatorname{add} M$-approximation of X. The exact sequence in (3) is called the connecting sequence.

The following is easily deduced.

Corollary 1.3. *Let M be an almost complete partial tilting module. Then M admits a source complement X_0. More precisely, for each complement Y to M with $Y \in \operatorname{fac} M$ there exists a long exact sequence*

$$0 \to X_0 \to M^1 \xrightarrow{f_1} M^2 \to \cdots \to M^{r-1} \xrightarrow{f_{r-1}} M^r \xrightarrow{f_r} X_r = Y \to 0$$

with $\ker f_i = X_{i-1}$ for $1 \leqslant i \leqslant r$, $M^i \in \operatorname{add} M$ for $1 \leqslant i \leqslant r$ and $X_0, \ldots, X_r$ complements to M where $r \leqslant \operatorname{pd}_\Lambda Y$.

In the following, we will always denote by X_0 the source complement to an almost complete partial tilting module M. The finitistic dimension of Λ is defined as

$$\operatorname{fd}(\Lambda) = \sup\{\operatorname{pd}_\Lambda X \mid X \in \mathcal{P}^{<\infty}(\Lambda)\}.$$

Theorem 1.4. *Let Λ be an Artin algebra. Let M be an almost complete partial tilting module. Then M has at most finitely many non-isomorphic complements if and only if M admits a sink complement. If there are $r+1$ non-isomorphic complements for some $r \geqslant 1$, then $r \leqslant \operatorname{fd}(\Lambda)$ and there exists a long exact sequence*

$$0 \to X_0 \to M^1 \xrightarrow{f_1} M^2 \to \cdots \to M^{r-1} \xrightarrow{f_{r-1}} M^r \xrightarrow{f_r} X_r \to 0$$

with $\ker f_i = X_{i-1}$ for $1 \leqslant i \leqslant r$, $M^i \in \operatorname{add} M$ for $1 \leqslant i \leqslant r$ and $X_0, \ldots, X_r$ complements to M. In particular, if $\operatorname{fd}(\Lambda) < \infty$, then M has finitely many complements.

Let M be an almost complete partial tilting module and X a complement to M. Then $X \simeq X_s$ for some s. Then we will say that the number of complements to M preceding X is $s+1$. If there is a finite number $r+1$ of complements to M, we say that the number of complements to M succeeding X is $r+1-s$. Otherwise, we will say that there are infinitely many complements to M succeeding X.

We will also point out the main connection to the generalised Nakayama conjecture but refer for the details to [HU3].

Theorem 1.5. *Let M be an almost complete partial tilting module with source complement X_0. Let $\Gamma = \operatorname{End}_\Lambda(M \oplus X_0)$. Let ${}_\Gamma S$ be the simple top of the indecomposable projective Γ-module $\operatorname{Hom}_\Lambda(M \oplus X_0, X_0)$. If M admits t complements different from X_0, then $\operatorname{Ext}^j_\Gamma(D(\Gamma_\Gamma), S) = 0$ for $0 \leqslant j < t$.*

2. Hasse quiver and consequences

In the introduction we have defined the partially ordered set $(\mathcal{T}_\Lambda, \leqslant)$ of tilting modules of an Artin algebra Λ. We start by observing some elementary properties. The assertion 2.1(a) is contained in [HU4].

Lemma 2.1. *Let Λ be an Artin algebra.*

(a) *$T \leqslant T'$ in $\mathcal{T}_\Lambda$ if and only if $T \in T'^{\perp}$.*
(b) *If $T \leqslant T'$ in $\mathcal{T}_\Lambda$ then $\operatorname{pd}_\Lambda T \geqslant \operatorname{pd}_\Lambda T'$.*

Proof. If $T \leqslant T'$ then $T \in T'^{\perp}$. Let $r = \operatorname{pd}_\Lambda T$. Then by the defining property of a tilting module there is an exact sequence

$$0 \to {}_\Lambda\Lambda \to T^0 \to \cdots \to T^s \to 0$$

with $T^i \in \operatorname{add} T$ for all $0 \leqslant i \leqslant s$. It is well known that we may choose such a sequence with $s = r$. Since $T \in T'^{\perp}$, it follows that $\operatorname{Ext}^{i+1}_\Lambda(T', {}_\Lambda\Lambda) = 0$ for $i \geqslant r$. Since $\operatorname{pd}_\Lambda T' < \infty$, we infer that $\operatorname{pd}_\Lambda T' \leqslant r$. $\square$

The following theorem and its corollary are contained in [HU4] and they are stated for the convenience of the reader. The theorem will be used in the next section.

Theorem 2.2. *Let $T, T' \in \mathcal{T}_\Lambda$ with $T \leqslant T'$ and $T \neq T'$. Then there exists $T'' \in \mathcal{T}_\Lambda$ with $T' \to T''$ in $\vec{\mathcal{K}}_\Lambda$ such that $T \leqslant T''$. In particular, $\vec{\mathcal{K}}_\Lambda$ is the Hasse quiver of $\mathcal{T}_\Lambda$.*

Corollary 2.3. *If $\vec{\mathcal{K}}_\Lambda$ has a finite connected component $\mathcal{C}$, then $\vec{\mathcal{K}}_\Lambda = \mathcal{C}$.*

We will deduce some further consequences.

Corollary 2.4. *If $\vec{\mathcal{K}}_\Lambda$ has a finite component $\mathcal{C}$, let T be the minimal element in $\mathcal{C}$. Let $d = \operatorname{pd}_\Lambda T$. Then for each $0 \leqslant i \leqslant d$ there exists a tilting module T_i with $\operatorname{pd}_\Lambda T_i = i$.*

Proof. By Corollary 2.3 we have that $\mathcal{C} = \vec{\mathcal{K}}_\Lambda$. If $T' \to T''$ is an arrow in $\vec{\mathcal{K}}_\Lambda$, we have by Lemma 2.1 that $\operatorname{pd}_\Lambda T'' \geqslant \operatorname{pd}_\Lambda T'$. By the definition of arrows in $\vec{\mathcal{K}}_\Lambda$, we actually have that

$$_\Lambda T' \leqslant \operatorname{pd}_\Lambda T'' \leqslant \operatorname{pd}_\Lambda T' + 1. \tag{$*$}$$

Since $\vec{\mathcal{K}}_\Lambda$ is finite there is a path in $\vec{\mathcal{K}}_\Lambda$ from the unique source ${}_\Lambda\Lambda$ of $\vec{\mathcal{K}}_\Lambda$ to the unique minimal element T of $\vec{\mathcal{K}}_\Lambda$ of the form

$${}_\Lambda\Lambda \to T_0 \to T_1 \to \cdots \to T_{s-1} \to T_s \to T.$$

Now $\operatorname{pd}_\Lambda \Lambda = 0$ and $\operatorname{pd}_\Lambda T = d$. Using the chain $(*)$ of inequalities shows the assertion. $\square$

The following is an immediate consequence of the previous corollary.

Corollary 2.5. *Let Λ be a representation-finite Artin algebra of finitistic dimension d. Then for each $0 \leqslant i \leqslant d$ there exists an indecomposable exceptional module E_i with $\operatorname{pd}_\Lambda E_i = i$.*

Proof. Let T be the sink in $\vec{\mathcal{K}}_\Lambda$. It follows from [HU2,HU4] that each Λ-module X with $\operatorname{pd} X < \infty$ satisfies $X \in \operatorname{sub} T$. Since the finitistic dimension of Λ is d there is X with $\operatorname{pd} X = d$. So let

$$0 \to X \to T' \to Y \to 0$$

be exact with $T' \in \operatorname{add} T$. Since $\operatorname{pd} X = d$ and $\operatorname{pd} Y \leqslant d$, we infer that $\operatorname{pd} T' = d$. So the assertion follows from Corollary 2.4. $\square$

We refer to [HU4] for a result about the existence of minimal elements in the partial order $\mathcal{T}_\Lambda$ which uses [HU2].

3. Local structure of the quiver for hereditary algebras

In the introduction we have defined the partially ordered set $(\mathcal{T}_\Lambda, \leqslant)$ and the quiver $\vec{\mathcal{K}}_\Lambda$ of tilting modules. In this section we will mainly be interested in the local structure of $\vec{\mathcal{K}}_\Lambda$ for hereditary algebras. So for the rest of this section, Λ denotes a hereditary Artin algebra.

Let $T \in \vec{\mathcal{K}}_\Lambda$. We denote by $s(T)$ (respectively $e(T)$) the number of arrows starting (respectively ending) at T in $\vec{\mathcal{K}}_\Lambda$ (see Fig. 1).

Lemma 3.1. *$s(T) + e(T) \leqslant \operatorname{rk} K_0(\Lambda) = n$.*

Proof. Let $T = \bigoplus_{i=1}^n T_i \in \vec{\mathcal{K}}_\Lambda$. For each $1 \leqslant i \leqslant n$, we consider the almost complete partial tilting module $T[i]$. For $T[i]$, we have an indecomposable complement T_i. If T_i is generated by $T[i]$, we obtain from Proposition 1.2 a complement X_i which is cogenerated by $T[i]$, hence an arrow from $T[i] \oplus X_i \to T$ in $\vec{\mathcal{K}}_\Lambda$. If T_i is cogenerated by $T[i]$, we obtain from Proposition 1.2 a complement Y_i which is generated by $T[i]$, hence an arrow from $T \to T[i] \oplus Y_i$ in $\vec{\mathcal{K}}_\Lambda$. Thus to show the assertion, it is enough to show that T_i cannot be both generated and cogenerated by $T[i]$. If T_i is both generated and cogenerated by $T[i]$, we obtain that $T[i]$ admits at least three non-isomorphic complements. Since Λ is hereditary, this contradicts Theorem 1.4. $\square$

We say that T is saturated if equality holds. Note that the proof of Lemma 3.1 shows that T is saturated if and only if $T[i]$ is faithful or equivalently sincere for each $1 \leqslant i \leqslant n$.

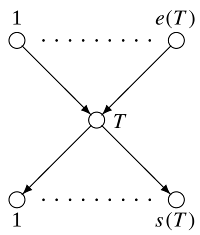

Fig. 1.

For a Λ-module X we denote by $\mathbf{dim}\, X \in K_0(\Lambda)$ its dimension vector which we will identify with the corresponding element in $\mathbb{Z}^n$.

Proposition 3.2. *$T \in \overrightarrow{\mathcal{K}}_\Lambda$ is saturated iff $(\mathbf{dim}\, T)_i \geqslant 2$ for all $1 \leqslant i \leqslant n$.*

Proof. If there is $1 \leqslant i \leqslant n$ with $(\mathbf{dim}\, T)_i = 1$, choose an indecomposable direct summand X of T with $(\mathbf{dim}\, X)_i = 1$. Let $T = M \oplus X$. Then $(\mathbf{dim}\, M)_i = 0$, so M is not sincere. Thus T is not saturated by the remark following the proof of Lemma 3.1.

Conversely, assume that $T \in \overrightarrow{\mathcal{K}}_\Lambda$ is not saturated. So there is an indecomposable direct summand X of $T = M \oplus X$ such that M is not sincere. So there exists an indecomposable injective Λ-module I such that $\mathrm{Hom}_\Lambda(M, I) = 0$. Let $\Gamma = \mathrm{End}_\Lambda\, T$. Then $\mathrm{Hom}_\Lambda(T, I)$ is an indecomposable Γ-module. Set $P_X = \mathrm{Hom}_\Lambda(T, X)$ and $S_X = \mathrm{top}\, P_X$. Now

$$\mathrm{Hom}_\Lambda(T, I) \simeq S_X^{\dim \mathrm{Hom}_\Lambda(T,I)} \simeq S_X^{\dim \mathrm{Hom}_\Lambda(X,I)}.$$

Since $\mathrm{Hom}_\Lambda(T, I)$ is indecomposable, we infer that $\dim \mathrm{Hom}_\Lambda(T, I) = 1$. Thus there is $1 \leqslant i \leqslant n$ with $(\mathbf{dim}\, T)_i = 1$. This finishes the proof. $\square$

We have the following consequence for the path algebra Λ of $\Delta = \mathbb{A}_n$ with linear orientation over a field k. There are no saturated vertices in $\overrightarrow{\mathcal{K}}_\Lambda$. In fact, it is well known that for a tilting module T over the path algebra of $\mathbb{A}_n$ with linear orientation the unique indecomposable projective–injective module is a direct summand and the direct sum of the other indecomposable summands is not sincere. Thus there is a coordinate of the dimension vector of T with value 1.

We will point out a different way of detecting arrows in $\overrightarrow{\mathcal{K}}_\Lambda$. For this, let $T \in \mathcal{T}_\Lambda$ and $\Gamma = \mathrm{End}\, T$. Let $(\mathcal{X}(T), \mathcal{Y}(T))$ be the torsion pair in $\mathrm{mod}\, \Gamma$ induced by T (for details, compare [HR]). Let $T = M \oplus X$ and let P_X, S_X be as above.

Proposition 3.3. *With the above notation the following hold.*

(1) *$S_X \in \mathcal{X}(T)$ if and only if $X \in \mathrm{fac}\, M$.*
(2) *$X \in \mathrm{sub}\, M$ if and only if $S_X \in \mathcal{Y}(T)$ and $T \otimes_\Gamma S_X$ is not injective.*

Proof. (1) If $X \in \mathrm{fac}\, M$, we consider an exact sequence $0 \to Y \to \widetilde{M} \to X \to 0$ with $\widetilde{M} \to X$ a minimal right $\mathrm{add}\, M$-approximation of X. Applying $\mathrm{Hom}_\Lambda(T, -)$ to this sequence yields an exact sequence of Γ-modules

$$\mathrm{Hom}_\Lambda\big(T, \widetilde{M}\big) \to P_X \to \mathrm{Ext}^1_\Lambda(T, Y) \to 0.$$

Now $\mathrm{Ext}^1_\Lambda(T, Y) \simeq S_X$ and $\mathrm{Ext}^1_\Lambda(T, Y) \in \mathcal{X}(T)$ by tilting theory, thus $S_X \in \mathcal{X}(T)$.

Conversely, assume that $S_X \in \mathcal{X}(T)$. Consider the exact sequence of Γ-modules

$$0 \to K_X \to P_X \to S_X \to 0.$$

Since $P_X \in \mathcal{Y}(T)$, we see that $K_X \in \mathcal{Y}(T)$, for $\mathcal{Y}(T)$ is a torsion-free class. Now $\mathrm{End}\, X$ is a division ring, since X is an indecomposable direct summand of a tilting module. By tilting theory, we infer that the same holds for $\operatorname{End} P_X$. Hence S_X is not a composition factor of K_X and so we see that $K_X \in \operatorname{fac}\operatorname{Hom}_\Lambda(T, M)$. But this implies that $X \in \operatorname{fac} M$.

(2) If $X \in \operatorname{sub} M$, we consider an exact sequence

$$0 \to X \to \widetilde{M} \to Y \to 0$$

with $X \to \widetilde{M}$ a minimal left $\operatorname{add} M$-approximation of X. Applying $\operatorname{Hom}_\Lambda(T, -)$ to this sequence yields an exact sequence of Γ-modules

$$0 \to P_X \to \operatorname{Hom}_\Lambda\big(T, \widetilde{M}\big) \to \operatorname{Hom}_\Lambda(T, Y) \to 0.$$

Applying $\operatorname{Hom}_\Gamma(-, S_X)$ to this exact sequence shows that $\operatorname{Ext}^1_\Gamma(\operatorname{Hom}_\Lambda(T, Y), S_X) \neq 0$. Since $\operatorname{Hom}_\Lambda(T, Y) \in \mathcal{Y}(T)$ and $(\mathcal{X}(T), \mathcal{Y}(T))$ is a split torsion pair, we infer that $S_X \in \mathcal{Y}(T)$. Since $X \in \operatorname{sub} M$, we have that M is sincere. Assume to the contrary that $T \otimes_\Gamma S_X = I$ is an injective Λ-module. Then $\operatorname{Hom}_\Lambda(T, I) = S_X$. Now $\operatorname{Hom}_\Lambda(X, I) = \operatorname{Hom}_\Lambda(P_X, S_X) \neq 0$ shows that $\operatorname{Hom}_\Lambda(M, I) = 0$. Thus M is not sincere, a contradiction.

Conversely, let $S_X \in \mathcal{Y}(T)$ and $T \otimes_\Gamma S_X$ not injective as Λ-module. Assume to the contrary that $X \notin \operatorname{sub} M$. Since $S_X \in \mathcal{Y}(T)$, we have by (1) that $X \notin \operatorname{fac} M$. Hence M is not sincere. So there exists an indecomposable injective Λ-module I with $\operatorname{Hom}_\Lambda(M, I) = 0$ and $\operatorname{Hom}_\Lambda(T, I) = S_X$. But then by tilting theory we have that $T \otimes_\Gamma S_X = I$, a contradiction. □

This yields a module theoretic characterisation of saturated vertices.

Corollary 3.4. *Let $T \in \vec{\mathcal{K}}_\Lambda$ and $\Gamma = \operatorname{End} T$. Then T is saturated if and only if for all simple Γ-modules $S \in \mathcal{Y}(T)$ the Λ-module $T \otimes_\Gamma S$ is not injective.*

Proof. Let $T \in \vec{\mathcal{K}}_\Lambda$ and $S = S_X \in \mathcal{Y}(T)$ be a simple Γ-module. Let $T = M \oplus X$. Since $S_X \in \mathcal{Y}(T)$, we have by Proposition 3.3(1) that $X \notin \operatorname{fac} M$. Since T is saturated, we have that $X \in \operatorname{sub} M$. But then by Proposition 3.3(2) we have that $T \otimes_\Gamma S_X$ is not injective as Λ-module.

Conversely, let X be an indecomposable direct summand of $T = M \oplus X$. If $S_X \in \mathcal{X}(T)$, we have by Proposition 3.3(1) that $X \in \operatorname{fac} M$. If $S_X \in \mathcal{Y}(T)$, we have by Proposition 3.3(2) that $X \in \operatorname{sub} M$ using the assumption that $T \otimes_\Gamma S$ is not injective as Λ-module. Thus T is saturated. □

The following theorem is contained in [U].

Theorem 3.5. *Each connected component of $\vec{\mathcal{K}}_\Lambda$ contains a non-saturated vertex.*

Remark 3.6. We conjecture that each connected component of $\vec{\mathcal{K}}_\Lambda$ contains only finitely many non-saturated vertices.

In the next section we will present a result about paths starting and ending in a given vertex $T \in \vec{\mathcal{K}}_\Lambda$. As a special case, we give here the result for neighbours. For a proof, we refer to the next section. For this let $0 \to {}_\Lambda\Lambda \to T^0 \to T^1 \to 0$ and $0 \to T_1 \to T_0 \to D\Lambda_\Lambda \to 0$ be minimal $\operatorname{add} T$-resolutions.

Proposition 3.7. *$s(T) = n - \delta(T_0) = \delta(T_1)$ and $e(T) = n - \delta(T^0) = \delta(T^1)$.*

Proof. The equalities $s(T) = \delta(T_1)$ and $e(T) = \delta(T^1)$ follow from the main theorem in the next section. The equalities $n - \delta(T_0) = \delta(T_1)$ and $n - \delta(T^0) = \delta(T^1)$ follow from Lemma 1.1 and its dual by observing that each indecomposable direct summand of T has to occur in the above resolutions. □

Corollary 3.8. *$T \in \vec{\mathcal{K}}_\Lambda$ is saturated if and only if $n = \delta(T^0) + \delta(T_0)$. In particular, ${}_\Lambda\Lambda$ is not saturated.*

Proof. This follows immediately from Proposition 3.7. □

We close this section by a result on regular tilting modules for wild hereditary algebras. In general, a regular tilting module will have neighbours in $\vec{\mathcal{K}}_\Lambda$ which have preprojective or preinjective indecomposable direct summands.

Proposition 3.9. *Let Λ be a wild hereditary algebra with at least three simple modules. Let $T \in \vec{\mathcal{K}}_\Lambda$ be regular. Then there exist $T', T'' \in \vec{\mathcal{K}}_\Lambda$ regular with $T' \to T \to T''$ in $\vec{\mathcal{K}}_\Lambda$.*

Proof. Let $T \in \vec{\mathcal{K}}_\Lambda$ be a regular tilting module. Then there is r_0 such that for all $r \geqslant r_0$ we have that $\operatorname{Ext}^1_\Lambda(T, \tau^r T) = 0$ (compare, for example, [K]). Thus $\tau^r T \leqslant T$ in $\mathcal{T}_\Lambda$ for all $r \geqslant r_0$. By Theorem 2.2 and the fact that T has only finitely many neighbours in $\vec{\mathcal{K}}_\Lambda$, there exists $T'' \in \vec{\mathcal{K}}_\Lambda$ and $T \to T''$ in $\vec{\mathcal{K}}_\Lambda$ such that $\tau^r T \leqslant T''$ for infinitely many $r \geqslant r_0$. Let $T = M \oplus X$ and $T'' = M \oplus Y$. The arrow in $\vec{\mathcal{K}}_\Lambda$ yields an exact sequence $0 \to X \to \widetilde{M} \to Y \to 0$ with $\widetilde{M} \in \operatorname{add} M$. Since T is regular, we see that Y is not preprojective. Let $r \geqslant r_0$ with $\tau^r T \leqslant T''$. Then $\operatorname{Ext}^1_\Lambda(Y, \tau^r T) = 0$, hence $\operatorname{Hom}_\Lambda(\tau^{r-1} T, Y) = 0$. Assume that Y is preinjective, so $Y = \tau^s I$ for some $s \geqslant 0$ and some indecomposable injective Λ-module. So $\operatorname{Hom}_\Lambda(\tau^{r-1-s} T, I) = 0$. But $\tau^t T$ is a tilting module for all t, hence sincere. This is a contradiction, so Y is regular, hence T'' is regular.

The existence of the arrow $T' \to T$ with T' regular is proved dually. □

4. Local structure of the quiver for arbitrary algebras

In this section, let Λ be an arbitrary Artin algebra. We will investigate here certain paths starting or ending in a given vertex $T \in \vec{\mathcal{K}}_\Lambda$. Before proving the main theorem in this section, we will need some preparation.

Lemma 4.1. *Let Λ be an Artin algebra and M an almost complete tilting module. Let X be a complement to M. Let $T = M \oplus X$. Let*

$$0 \to K \to V \to D(\Lambda_\Lambda) \to 0 \tag{$*$}$$

be the minimal right add T*-approximation of* $D(\Lambda_\Lambda)$*. Then* $V \in \operatorname{add} M$*, if and only if* $X \in \operatorname{sub} M$.

Proof. Let $X \in \operatorname{mod} \Lambda$ and $\mu : X \to D(\Lambda_\Lambda)^s$ be an injective map for some $s > 0$. Since $(*)$ is the minimal right add T-approximation of $D(\Lambda_\Lambda)$. we have that $\operatorname{Ext}^1_\Lambda(X, K) = 0$. This implies that μ factors over $V^s \to (D(\Lambda_\Lambda))^s \to 0$, so $X \in \operatorname{sub} V$. Thus by assumption $X \in \operatorname{sub} M$.

Conversely, assume that $X \in \operatorname{sub} M$. In particular, we have that M is faithful. Consider as above the connecting sequence $0 \to X \to \widetilde{M} \to Y \to 0$ with $\widetilde{M} \in \operatorname{add} M$. Moreover, let

$$0 \to K' \to M' \to D(\Lambda_\Lambda) \to 0 \tag{$**$}$$

be the minimal right add M-approximation of $D(\Lambda_\Lambda)$. Then we have $\operatorname{Ext}^i_\Lambda(X, K') = \operatorname{Ext}^{i+1}_\Lambda(Y, K') = \operatorname{Ext}^{i+1}_\Lambda(Y, M') = 0$, for $M \oplus Y$ is a tilting module. Thus $(**)$ is the minimal right add T-approximation of $D(\Lambda_\Lambda)$. So $V \in \operatorname{add} M$. This proves the assertion.

We will also use the dual statement which we state without proof. □

Lemma 4.2. *Let* Λ *be an Artin algebra and* M *an almost complete tilting module. Let* X *be a complement to* M*. Let* $T = M \oplus X$*. Let*

$$0 \to {}_\Lambda\Lambda \to W \to Q \to 0 \tag{$*$}$$

be the minimal left add T*-approximation of* ${}_\Lambda\Lambda$*. Then* $W \in \operatorname{add} M$ *if and only if* $X \in \operatorname{fac} M$.

We will now fix some more notation. For $T \in \vec{\mathcal{K}}_\Lambda$, consider the following two minimal add T-resolutions:

$$0 \to {}_\Lambda\Lambda \to T^0 \to \cdots \to T^r \to 0, \tag{$+$}$$

$$\cdots \to T_s \to \cdots \to T_0 \to D\Lambda_\Lambda \to 0. \tag{$++$}$$

Let X be an indecomposable direct summand of T. We choose $i(X)$ minimal such that X is a direct summand of $T^{i(X)}$. Note that each indecomposable direct summand X of T has to occur in $(+)$, so $i(X)$ is well defined. If X occurs in $(++)$, we choose $j(X)$ minimal such that X is a direct summand of $T_{j(X)}$. Otherwise, we set $j(X) = \infty$.

Note that the validity of the generalised Nakayama conjecture will imply that $j(X) < \infty$. For this, compare [HU3] and Theorem 1.5.

Lemma 4.3. *Let* T *be a tilting module and let* X *be an indecomposable direct summand of* $T = M \oplus X$*. Then the number of complements to* M *preceding* X *is* $i(X) + 1$.

Proof. Let

$$0 \to {}_\Lambda\Lambda \to T^0 \to \cdots \to T^r \to 0$$

be the minimal add T-resolution of ${}_\Lambda\Lambda$. We will show the assertion by induction on $i(X)$. If $i(X)=0$, then $X \in \operatorname{add} T^0$. By Lemma 4.2 we infer that $X \notin \operatorname{fac} M$, hence X is the source complement to M. Since in this case there is precisely one complement to M preceding X, the assertion holds. If $i(X)>0$, let $X = X_s$. We consider the long exact sequence of complements to M preceding X_s:

$$0 \to X_0 \to M^0 \to \cdots \to M^{s-1} \to X_s \to 0$$

with $M^i \in \operatorname{add} M$ for $0 \leqslant i \leqslant s-1$. Moreover, let $0 \to X_{s-1} \to M^{s-1} \to X_s \to 0$ be the connecting sequence. Let $T' = M \oplus X_{s-1}$. By induction, we have that $s = i(X_{s-1})+1$. We consider the start of the minimal add T'-resolution of ${}_\Lambda\Lambda$:

$$0 \to {}_\Lambda\Lambda \to T'^0 \to \cdots \to T'^s. \tag{$*$}$$

By construction, we have that $X_{s-1} \in \operatorname{add} T'^{s-1}$ and $T'^i \in \operatorname{add} M$ for $0 \leqslant i < s-1$. From $(*)$ consider $0 \to Q^{s-2} \to T'^{s-1} \to Q^{s-1} \to 0$. Since $X_{s-1} \in \operatorname{sub} M$, we have that $Q^{s-2} \in \operatorname{sub} M$. Let

$$0 \to Q^{s-2} \to \widetilde{M} \to Q'^{s-1} \to 0 \tag{$**$}$$

be the minimal left add M-approximation of Q^{s-2}. Now we have $\operatorname{Ext}^i_\Lambda(Q'^{s-1}, X_s) = \operatorname{Ext}^{i+1}_\Lambda(Q'^{s-1}, X_{s-1}) = \operatorname{Ext}^i_\Lambda(Q^{s-2}, X_{s-1}) = 0$. Thus $(**)$ is also the minimal left add T-approximation of Q^{s-2}. So we have that $s \leqslant i(X_s)$. Assume that $s < i(X_s)$. Then $Q'^{s-1} \in \operatorname{sub} M$ and the minimal left add M-approximation $0 \to Q'^{s-1} \to M' \to Q^s \to 0$ of Q'^{s-1} is also the minimal left add T-approximation of Q'^{s-1}. Thus $\operatorname{Ext}^i_\Lambda(Q^s, X_s) = 0$. But $0 = \operatorname{Ext}^i_\Lambda(Q^s, X_s) = \operatorname{Ext}^{i+1}_\Lambda(Q^s, X_{s-1}) = \operatorname{Ext}^i_\Lambda(Q'^{s-1}, X_{s-1})$ shows that $(**)$ is also the minimal left add T'-approximation, in contrast to $i(X_{s-1}) = s-1$. This finishes the proof of the assertion. □

We will use the following dual lemma without proof.

Lemma 4.4. *Let T be a tilting module and let X be an indecomposable direct summand of $T = M \oplus X$. Then the number of complements to M succeeding X is $j(X)+1$.*

Theorem 4.5. *For each indecomposable direct summand X of T, there exists a path $w(X)$ in $\vec{\mathcal{K}}_\Lambda$ of length $i(X)$ ending at T and a path $u(X)$ in $\vec{\mathcal{K}}_\Lambda$ of length $j(X)$ starting in T. These paths are pairwise disjoint.*

Proof. By Lemmas 4.3 and 4.4, the assertions on the paths follow directly from the definition of arrows in $\vec{\mathcal{K}}_\Lambda$. So it remains to show that the paths are pairwise disjoint. Let $T = M \oplus X = N \oplus Y \in \vec{\mathcal{K}}_\Lambda$ with X not isomorphic to Y. So X is an indecomposable direct summand of N and Y is an indecomposable direct summand of M. Let

$$T_{i(X)} \to \cdots \to T_1 \to M \oplus X \to T^1 \to \cdots \to T^{j(X)}$$

and

$$\overline{T}_{i(Y)} \to \cdots \to \overline{T}_1 \to N \oplus Y \to \overline{T}^1 \to \cdots \to \overline{T}^{j(Y)}$$

be the paths given by Lemmas 4.3 and 4.4. Suppose that these paths are not disjoint. Since $\vec{\mathcal{K}}_\Lambda$ does not contain oriented cycles, there exist i and j such that $T_i = \overline{T}_j$ or $T^i = \overline{T}^j$. Assume that $T_i = \overline{T}_j$. By construction, we have that $T_i = M \oplus X_i$ for some complement X_i to M and that $\overline{T}_j = N \oplus Y_j$ for some complement Y_j to N. Since X is not isomorphic to Y, the same holds for X_i and Y_j. So X_i is a direct summand of N. Thus $\mathrm{Ext}^t_\Lambda(X, X_i) = 0$ for all $t > 0$. But by Theorem 1.4, we have that $\mathrm{Ext}^i_\Lambda(X, X_i) \neq 0$, a contradiction. The case that $T^i = \overline{T}^j$ can be ruled out similarly. □

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